

# ON THE COHOMOLOGICAL EQUATION FOR NILFLOWS

LIVIO FLAMINIO AND GIOVANNI FORNI

**ABSTRACT.** Let  $X$  be a vector field on a compact connected manifold  $M$ . An important question in dynamical systems is to know when a function  $g : M \rightarrow \mathbb{R}$  is a coboundary for the flow generated by  $X$ , i.e. when there exists a function  $f : M \rightarrow \mathbb{R}$  such that  $Xf = g$ . In this article we investigate this question for nilflows on nilmanifolds. We show that there exists countably many independent Schwartz distributions  $D_n$  such that any sufficiently smooth function  $g$  is a coboundary iff it belongs to the kernel of all the distributions  $D_n$ .

## 1. INTRODUCTION

**1.1. The problem.** For a detailed discussion of the cohomological equation for flows and transformations in ergodic theory, we refer the reader to [Kat03]. Here we limit ourselves to a brief, self-contained introduction.

Let  $X$  be a smooth vector field on a connected compact manifold  $\mathcal{M}$  and let  $(\phi_X^t)_{t \in \mathbb{R}}$  be the flow generated by  $X$  on  $\mathcal{M}$ . Many problems in dynamics and ergodic theory (see loc. cit.) can be reconducted to the study of the *cohomological equation*

$$(1) \quad Xu = f,$$

i.e. the problem of finding a function  $u$  on  $\mathcal{M}$  whose Lie derivative  $Xu$  in the direction of  $X$  is a given function  $f$  on  $\mathcal{M}$ . Clearly a continuous solution  $u$  of the equation (1) is only determined up to function constant along the orbits. In addition, given the value of  $u$  at one point  $p \in \mathcal{M}$ , the solution  $u$  is uniquely determined on the whole orbit of  $p$  under the flow  $(\phi_X^t)_{t \in \mathbb{R}}$ . In fact, the recurrence properties of flow  $(\phi_X^t)_{t \in \mathbb{R}}$  may as well forbid the existence of measurable solutions to (1). Hence the subtlety of the problem lies entirely in the fact that, given  $f$  in a certain regularity class, we may or may not have solutions  $u$  in some other regularity class.

For example, observe that if the cohomological equation (1) admits a continuous solution  $u$  then

$$(2) \quad \mu(f) = 0 \text{ for all } X\text{-invariant Borel measures } \mu \in C^*(\mathcal{M}).$$

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Thus invariant measures are obstructions to the existence of solutions of (1) in the continuous class. Livshitz's celebrated theorem states that if  $X$  is an Anosov vector field and  $f$  is Hölder continuous of exponent  $\alpha \in (0, 1)$ , then the necessary condition (2) is also sufficient and that, in this case, the solution  $u$  is actually Hölder continuous of exponent  $\alpha$ . (This theorem has been generalized to other hyperbolic dynamical systems and to smoother classes of functions, see for example [KH95]).

For uniquely ergodic flows, i.e. for flows admitting a unique invariant measure, one may wonder whether the above condition (2) is sufficient. It is well known that, for  $X$  a constant vector field on a torus,

- if  $X$  is Diophantine, for any  $C^\infty$  function  $f$  of average zero there exists a  $C^\infty$  solution  $u$  of equation (1);
- if  $X$  is Liouvillean, then there exists some  $C^\infty$  function  $f$  of average zero for which there is no measurable weak<sup>1</sup> solution  $u$  of equation (1).

In general, all invariant distributions (in the sense of Schwartz), which are not necessarily signed measures, are obstructions to the existence of smooth solutions of the cohomological equation.

A distribution  $D \in \mathcal{D}'(\mathcal{M})$  is called  $X$ -invariant if  $XD = 0$  in the distributional sense, that is, if  $D(X\phi) = 0$  for all  $\phi \in C^\infty(\mathcal{M})$ . Let  $\mathcal{I}_X^\infty(\mathcal{M})$  be the space of all  $X$ -invariant distributions in  $\mathcal{D}'(\mathcal{M})$ .

By definition, if  $u \in C^\infty(\mathcal{M})$  is a solution of (1), we must have  $D(f) = 0$  for all  $D \in \mathcal{I}_X^\infty(\mathcal{M})$ . By the same token, if  $u$  is a solution of (1) which is  $(\alpha + 1)$ -times differentiable, then we must have  $D(f) = 0$  for all  $X$ -invariant distributions  $D \in \mathcal{D}'(\mathcal{M})$  of order  $\alpha > 0$ .

Let  $W^\alpha(\mathcal{M})$  be the Sobolev space of square-integrable functions with square-integrable (weak) derivatives up to order  $\alpha > 0$  and let  $\mathcal{I}_X^\alpha(\mathcal{M}) \subset \mathcal{I}_X^\infty(\mathcal{M})$  be the space of  $X$ -invariant distributions of Sobolev order  $\alpha > 0$ , i.e. which extend continuously to the Sobolev space  $W^\alpha(\mathcal{M})$ .

If  $u \in W^{\alpha+1}(\mathcal{M})$  is a solution of (1), then  $D(f) = 0$  for all  $D \in \mathcal{I}_X^\alpha(\mathcal{M})$ . It is then natural to ask if and when invariant distributions are a complete set of obstructions to the existence of differentiable solutions, i.e. when the condition

$$(3) \quad f \in W^\alpha(\mathcal{M}) \quad \text{and} \quad D(f) = 0 \quad \text{for all } D \in \mathcal{I}_X^\alpha(\mathcal{M}),$$

is sufficient to guarantee the existence of a solution  $u$  satisfying some regularity properties (for example  $u \in W^{\alpha-k}(\mathcal{M})$  for some  $k \in \mathbb{R}^+$ ).

The main results of this paper is that this is the case for all *Diophantine* nilflows on compact nilmanifolds.

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<sup>1</sup>that is  $(u, X\phi) = -(f, \phi)$  for all  $\phi \in C^\infty(\mathcal{M})$

**1.2. Main result.** Let  $\mathfrak{n}$  be a  $k$ -step nilpotent real Lie algebra ( $k \geq 2$ ) with a minimal set of generators  $\mathcal{E} := \{E_1, \dots, E_n\} \subset \mathfrak{n}$ . Let  $\mathfrak{n}_j$ ,  $j = 1, \dots, k$ , denote the *descending central series* of  $\mathfrak{n}$ :

$$(4) \quad \mathfrak{n}_1 = \mathfrak{n}, \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}], \dots, \mathfrak{n}_j = [\mathfrak{n}_{j-1}, \mathfrak{n}], \dots, \mathfrak{n}_k \subset Z(\mathfrak{n}),$$

where  $Z(\mathfrak{n})$  is the center of  $\mathfrak{n}$ .

Let  $N$  be the connected and simply connected nilpotent Lie group with Lie algebra  $\mathfrak{n}$ . The corresponding Lie subgroups  $N_j = \exp \mathfrak{n}_j = [N_{j-1}, N]$  form the descending central series of  $N$ .

Let  $\Gamma$  be a lattice in  $N$ . It exists if and only if  $N$  admits rational structure constants (see, for example, [Rag72, CG90]).

A (compact) *nilmanifold* is a by definition a quotient manifold  $\mathcal{N} := \Gamma \backslash N$  with  $N$  a nilpotent Lie group and  $\Gamma \subset N$  a lattice.

On a nilmanifold  $\mathcal{N} = \Gamma \backslash N$ , the group  $N$  acts on the right transitively by right multiplication. By definition, the *nilflow*  $(\phi_X^t)_{t \in \mathbb{R}}$  generated by  $X \in \mathfrak{n}$  is the flow obtained by restriction of this action to the one-parameter subgroups  $(\exp tX)_{t \in \mathbb{R}}$  of  $N$ :

$$(5) \quad \phi_W^t(\Gamma x) = \Gamma x \exp(tX).$$

It is plain that nilflows on  $\Gamma \backslash N$  preserve the probability measure on  $\Gamma \backslash N$  given locally by the Haar measure. To simplify the notation, the vector field on  $\Gamma \backslash N$  generating the flow  $(\phi_X^t)_{t \in \mathbb{R}}$  will also be indicated by  $X$ .

Every nilmanifold is a fiber bundle over a torus. In fact, the group  $\overline{N} = N/[N, N]$  is abelian, connected and simply connected, hence isomorphic to  $\mathbb{R}^n$  and  $\overline{\Gamma} = \Gamma/[\Gamma, \Gamma]$  is a lattice in  $\overline{N}$ . Thus we have a natural projection

$$(6) \quad p : \Gamma \backslash N \rightarrow \overline{\Gamma} \backslash \overline{N}$$

over a torus of dimension  $n$ .

We recall the following:

**Theorem 1.1** ([Gre61], [AGH63]). *The following properties are equivalent.*

- (1) *The nilflow  $((\phi_X^t)_{t \in \mathbb{R}}, \mu)$  on  $\Gamma \backslash N$  is ergodic.*
- (2) *The nilflow  $(\phi_X^t)_{t \in \mathbb{R}}$  on  $\Gamma \backslash N$  is uniquely ergodic.*
- (3) *The nilflow  $(\phi_X^t)_{t \in \mathbb{R}}$  on  $\Gamma \backslash N$  is minimal.*
- (4) *The projected flow  $(\psi_X^t)_{t \in \mathbb{R}}$  on  $\overline{\Gamma} \backslash \overline{N} \approx \mathbb{T}^n$  is an irrational linear flow on  $\mathbb{T}^n$ , hence it is (uniquely) ergodic and minimal.*

The “irrationality” condition above in Theorem 1.1, (4), refers to the rational structure determined by the lattice  $\overline{\Gamma}$ . Namely, if the generators  $\{E_1, \dots, E_n\}$  of  $\mathfrak{n}$  are chosen so that the elements  $\{\exp E_1, \dots, \exp E_n\}$  project onto generators  $\{\exp \overline{E}_1, \dots, \exp \overline{E}_n\}$  of  $\overline{\Gamma}$ , then there exists a vector  $\Omega_X := (\omega_1(X), \dots, \omega_n(X)) \in \mathbb{R}^n$  such that

$$(7) \quad \bar{X} = \omega_1(X) \overline{E}_1 + \dots + \omega_n(X) \overline{E}_n$$

and the condition means that  $\omega_1(X), \dots, \omega_n(X)$  are linearly independent over  $\mathbb{Q}$ . We shall call such an element  $X \in \mathfrak{n}$  *irrational (with respect to  $\Gamma$ )*.

We have:

**Theorem 1.2.** *Let  $\mathcal{N} = \Gamma \backslash N$  be any compact nilmanifold obtained as a quotient of a  $k$ -step nilpotent ( $k \geq 2$ ) connected, simply connected Lie group  $N$  by a lattice  $\Gamma \subset N$ . For any irrational  $X \in \mathfrak{n}$ , the space of  $X$ -invariant distributions  $\mathcal{I}_X^\infty(\mathcal{N})$  has countable dimension. In fact,  $\mathcal{I}_X^\infty(\mathcal{N})$  admits a countable basis  $\mathcal{B}_X$  of invariant distributions of Sobolev order  $1/2$ , in the sense that  $\mathcal{B}_X \subset \mathcal{I}_X^\alpha(\mathcal{N})$  for any  $\alpha > 1/2$ .*

We shall say that  $X \in \mathfrak{n}$  is *Diophantine (with respect to  $\Gamma$ ) of exponent  $\tau \geq 0$*  if the projection  $\bar{X}$  of  $X$  in  $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$  is Diophantine of exponent  $\tau \geq 0$  for the lattice  $\bar{\Gamma}$  in the standard sense; namely, if there exists a constant  $K > 0$  such that, for all  $M := (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ ,

$$(8) \quad |\langle M, \Omega_X \rangle| := \left| \sum_{i=1}^n m_i \omega_i(X) \right| \geq \frac{K}{|M|^{n-1+\tau}}.$$

The subset of Diophantine elements  $X \in \mathfrak{n}$  of exponent  $\tau \geq 0$  will be denoted by  $\text{DC}_\tau(\mathfrak{n})$ . The subset  $\text{DC}(\mathfrak{n}) := \cup_\tau \text{DC}_\tau(\mathfrak{n})$  of all Diophantine elements has full Lebesgue measure in the Lie algebra  $\mathfrak{n}$ .

Let  $W_0^\alpha(\mathcal{N}, X) := W^\alpha(\mathcal{N}) \cap \ker \mathcal{I}_X^\alpha(\mathcal{N})$  for any  $\alpha > 1/2$ . We then have:

**Theorem 1.3.** *Let  $\mathcal{N}$ ,  $\Gamma$ ,  $N$  be as above. If  $X \in \text{DC}_\tau(\mathfrak{n})$  the following holds.*

- (1) *If  $\alpha > n + \tau - 1/2$  and  $\beta < -1/2$ , there exists a linear bounded operator  $G_X : W^\alpha(\mathcal{N}) \rightarrow W^\beta(\mathcal{N})$  such that, for all  $f \in W^\alpha(\mathcal{N})$ , the distribution  $u = G_X f$  is a weak solution of cohomological equation (1).*
- (2) *If  $\alpha > n + \tau$  and  $\beta < [\alpha - (n + \tau)][(n + \tau)k + 1]^{-1}$ , there exists a linear bounded operator  $G_X : W_0^\alpha(\mathcal{N}, X) \rightarrow W^\beta(\mathcal{N})$ , such that, for all  $f \in W^\alpha(\mathcal{N})$ , the function  $u = G_X f$  is a solution of cohomological equation (1).*

**1.3. Motivations and applications.** We have already mentioned that the cohomological problem for a flow (or a transformation) enters in many problems in dynamical systems (KAM, time-changes, etc.). We recall the following definitions:

**Definition 1.4.** *Let  $\mathcal{M}$  be a compact connected manifold. A smooth vector field  $X$  on  $\mathcal{M}$  is*

- (a) *globally hypoelliptic (GH), if  $Xu \in C^\infty(\mathcal{M})$  implies  $u \in C^\infty(\mathcal{M})$  for any distribution  $u \in \mathcal{D}'(\mathcal{M})$ ;*

- (b) cohomology free (CF), if for all  $f \in C^\infty(\mathcal{M})$  there exists a constant  $c(f) \in \mathbb{C}$  and  $u \in C^\infty(\mathcal{M})$  such that

$$Xu = f - c(f).$$

In [GW73] Greenfield and Wallach showed that, if a vector field  $X$  on  $\mathcal{M}$  is globally hypoelliptic, then there exists an invariant volume form  $\omega$  and the space of  $X$ -invariant distributions is reduced to the line  $\mathbb{C}\omega$  (in particular the  $X$ -flow is uniquely ergodic). The same is true, by a simple exercise, if  $X$  is cohomology free. In fact, if  $X$  is cohomology free, then it is globally hypoelliptic.

**Question 1.** *Is a globally hypoelliptic vector field on a compact connected manifold also cohomology free?*

**Definition 1.5** ([Kat01, Kat03]). *A vector field  $X$  on a compact connected manifold  $\mathcal{M}$  is said  $C^\infty$ -stable if the subspace  $\{Xu \mid u \in C^\infty(\mathcal{M})\}$  is closed in  $C^\infty(\mathcal{M})$ .*

It is plain that for  $C^\infty$ -stable vector fields the answer to the above question is positive. Absence of stability is related to fast approximation by periodic systems: the major example of  $C^\infty$ -unstable are Liouvillean constant vector fields on tori (loc.cit.). In [GW73], Greenfield and Wallach showed that if a Killing vector field of a Riemannian metric on  $\mathcal{M}$  is globally hypoelliptic, then the manifold  $\mathcal{M}$  is a torus and that on the flat 2-torus any Killing globally hypoelliptic vector field must be a constant Diophantine vector field. Motivated by these results they conjectured:

**Conjecture 1** ([GW73]). *If a compact, connected manifold  $\mathcal{M}$  admits a globally hypoelliptic vector field  $X$  then  $\mathcal{M}$  is a torus and  $X$  is smoothly conjugate to a constant Diophantine vector field.*

In fact, if  $\mathcal{M}$  is a torus, by [CC00] any globally hypoelliptic vector field  $X$  is smoothly conjugate to a constant Diophantine vector field.

A related conjecture due to A. Katok is the following:

**Conjecture 2** ([Kat01, Hur85, Kat03]). *If a compact, connected manifold  $\mathcal{M}$  admits a cohomology free vector field  $X$  then  $\mathcal{M}$  is a torus and  $X$  is smoothly conjugate to a constant Diophantine vector field.*

In support of Katok's conjecture Federico and Jana Rodriguez Hertz have recently proved in [RHRH05] that any (compact, connected) manifold  $\mathcal{M}$  admitting a cohomology free vector field fibres over a torus of dimension equal to the first Betti number of  $\mathcal{M}$ .

Theorems 1.2 and 1.3 immediately imply that, within the class of nilflows, both conjectures are true. As remarked by the Rodriguez Hertz's in

[RHRH05], nilflows are the simplest class of flows for which their theorem yields no non-trivial information.

Another motivation for the study of cohomological equations and invariant distributions comes from “renormalization dynamics” (see [For02a]). In fact, whenever we can define, for a family of “parabolic flows”, an effective renormalization dynamics on a suitable moduli space, the analysis of the action of the renormalization on the bundle of invariant distributions over the moduli space may allow to determine the asymptotics of the ergodic averages for the typical flow in the given family. This program was carried out in [For02b] for conservative flows on (higher genus) surfaces (related to interval exchange transformations), in [FF03b] for horocycle flows on hyperbolic surfaces and in [FF03a] for nilflows on the 3-dimensional Heisenberg group. These results have shown that the phenomenon of polynomial deviations of ergodic averages for interval exchange transformations, discovered by A. Zorich [Zor94], [Zor96], [Zor97] and rigorously proved in [For02b], is shared by other fundamental examples of parabolic flows. In principle, it should be possible to carry out this program by purely dynamical methods based on renormalization. In fact, in the case of interval exchange transformations, Marmi, Moussa and Yoccoz [MMY05] have been able to replace the harmonic analysis methods applied by the second author in his study of the cohomological equation [For97] by a renormalization approach (based on the Rauzy-Veech-Zorich induction). However, no renormalization dynamics is currently available for general nilflows.

## 2. IRREDUCIBLE UNITARY REPRESENTATIONS

**2.1. Kirillov’s Theory.** Let  $N$  be the connected and simply connected nilpotent Lie group with Lie algebra  $\mathfrak{n}$ . By Kirillov theory, all the irreducible unitary representation of  $N$  are parametrized by the *coadjoint orbits*  $\mathcal{O} \subset \mathfrak{n}^*$ , i.e. by the orbits of the *coadjoint action* of  $N$  on  $\mathfrak{n}^*$  defined by

$$(9) \quad \text{Ad}^*(g)\lambda = \lambda \circ \text{Ad}(g^{-1}) \quad g \in N, \lambda \in \mathfrak{n}^*.$$

For  $\lambda \in \mathfrak{n}^*$ , the skew-symmetric bilinear form

$$(10) \quad B_\lambda(X, Y) = \lambda([X, Y])$$

has a *radical*  $\mathfrak{r}_\lambda$  which coincides with the Lie subalgebra of the subgroup of  $N$  stabilizing  $\lambda$ ; thus the form (10) is non-degenerate on the tangent space to the orbit  $\mathcal{O} \subset \mathfrak{n}^*$  of  $\lambda$  and defines a symplectic form on  $\mathcal{O}$ .

A *polarizing (or maximal subordinate) subalgebra* for  $\lambda$  is a maximal isotropic subspace  $\mathfrak{m} \subset \mathfrak{n}$  for the form  $B_\lambda$  which is also a subalgebra of  $\mathfrak{n}$ . In particular any polarizing subalgebra for the linear form  $\lambda$  contains the radical  $\mathfrak{r}_\lambda$ . If  $\mathfrak{m}$  is a polarizing subalgebra for the linear form  $\lambda$ , the map

$$\exp T \mapsto \exp 2\pi i \lambda(T), \quad T \in \mathfrak{m},$$

yields a one-dimensional representation, which we denote by  $\exp 2\pi i\lambda$ , of the subgroup  $M = \exp \mathfrak{m} \subset N$ .

To a pair  $\Lambda := (\lambda, \mathfrak{m})$  formed by a linear form  $\lambda \in \mathfrak{n}^*$  and a polarizing subalgebra  $\mathfrak{m}$  for  $\lambda$ , we associate the unitary representation

$$(11) \quad \pi_\Lambda = \text{Ind}_{\exp \mathfrak{m}}^N(\exp 2\pi i\lambda).$$

These unitary representations are irreducible; up to unitary equivalence, all unitary irreducible representations of  $N$  are obtained in this way. Furthermore, two pairs  $\Lambda := (\lambda, \mathfrak{m})$  and  $\Lambda' := (\lambda', \mathfrak{m}')$  yield unitarily equivalent representations  $\pi_\Lambda$  and  $\pi_{\Lambda'}$  if and only if  $\lambda$  and  $\lambda'$  belong to the same coadjoint orbit  $\mathcal{O} \subset \mathfrak{n}^*$ .

The unitary equivalence class of the representations of the group  $N$  determined by the coadjoint orbit  $\mathcal{O}$  will be denoted by  $\Pi_{\mathcal{O}}$ , while we set

$$\Pi_\lambda = \{\pi_\Lambda \mid \Lambda = (\lambda, \mathfrak{m}), \text{ with } \mathfrak{m} \text{ polarizing subalgebra for } \lambda\}.$$

**Definition 2.1.** *Let  $N$  be a connected, simply connected Lie group of Lie algebra  $\mathfrak{n}$ . The derived representation  $\pi_*$  of a unitary representation  $\pi$  of  $N$  on a Hilbert space  $H_\pi$  is the Lie algebra representation of  $\mathfrak{n}$  on  $H_\pi$  defined as follows. For every  $X \in \mathfrak{n}$ ,*

$$(12) \quad \pi_*(X) := \text{strong-}\lim_{t \rightarrow 0} (\pi(\exp tX) - I)/t.$$

It can be proved that the derived representation  $\pi_*$  of the Lie algebra  $\mathfrak{n}$  on  $H_\pi$  is *essentially skew-adjoint* in following sense. For all  $X \in \mathfrak{n}$ , the linear operators  $\pi_*(X)$  are essentially skew-adjoint with common invariant core the subspace of  $C^\infty(H_\pi) \subset H_\pi$  of  $C^\infty$ -vectors in  $H_\pi$ . We recall that a vector  $v \in H_\pi$  is  $C^\infty$  for the representation  $\pi$  if the function  $g \in N \mapsto \pi(g)v \in H_\pi$  is of class  $C^\infty$  as a function on  $N$  (with values in a Hilbert space).

Suppose that  $N$  is the semi-direct product  $A \ltimes N'$  of a normal subgroup  $N'$  and an abelian group  $A$  and that  $\pi'$  is a unitary irreducible representation of  $N'$  on an Hilbert space  $H'$ ; then the derived representation  $\pi_*$  of the induced representation  $\pi = \text{Ind}_{N'}^N(\pi')$  has a simple description in terms of  $\pi'_*$ : in fact, up to unitary equivalence,  $H_\pi \approx L^2(A, H')$ ; furthermore the subgroup  $A$  acts by translations, and hence for  $f \in L^2(A, H')$  we have

$$(13) \quad (\pi_*(X)f)(a) = \frac{d}{dt} f(a \exp tX)|_{t=0}, \quad a \in A, X \in \mathfrak{a} := \text{Lie}(A);$$

the infinitesimal action of  $N'$  is a pointwise action given explicitly by:

$$(14) \quad (\pi_*(Y)f)(a) = \pi'_*(\text{Ad}(a)Y)(f(a)), \quad a \in A, Y \in \mathfrak{n}' := \text{Lie}(N').$$

Since, for any  $a \in A$ , the operator  $\text{Ad}(a)$  acting on  $\mathfrak{n}'$  is unipotent, the right-hand side in the formula above is a polynomial function in the variable

$a \in A$ . In fact, if  $\ell$  is the degree of nilpotency of  $\text{ad}(X)$  and  $a = \exp X$ , we have, for all  $Y \in \mathfrak{n}' = \text{Lie}(N')$ ,

$$(15) \quad (\pi_*(Y)f)(a) = \sum_{j=0}^{\ell} \frac{1}{j!} \pi'_* (\text{ad}(X)^j Y)(f(a)).$$

**2.2. Coadjoint orbits of maximal rank.** Let  $\mathfrak{n}$  be a  $k$ -step nilpotent real Lie algebra on  $n$  generators  $E_1, \dots, E_n$ . Let  $\mathfrak{n}_j$ ,  $j = 1, \dots, k$ , denote the *descending central series* of  $\mathfrak{n}$ :

$$\mathfrak{n}_1 = \mathfrak{n}, \mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}], \dots, \mathfrak{n}_j = [\mathfrak{n}_{j-1}, \mathfrak{n}], \dots, \mathfrak{n}_k \subset Z(\mathfrak{n}).$$

In this section we characterize coadjoint orbits that correspond to unitary representations which do not factor through the quotient  $N/\exp \mathfrak{n}_k$ . Such coadjoint orbits and the induced unitary representations will be called of *maximal rank*. Since the Lie group  $N/\exp \mathfrak{n}_k$  is  $(k-1)$ -step nilpotent, the analysis of unitary representations of maximal rank is sufficient to treat by induction all unitary representations of a given  $k$ -step nilpotent Lie group.

We shall make the fundamental assumption that the coadjoint orbit  $\mathcal{O} \subset \mathfrak{n}^*$  has maximal rank (see below). Before stating this assumption, a few lemmas and definitions.

Let

$$(16) \quad \begin{aligned} \mathfrak{n}_{k-1}^\perp(\lambda) &= \{X \in \mathfrak{n} \mid B_\lambda(X, \mathfrak{n}_{k-1}) = 0\}, \\ \mathfrak{n}_{k-1}^\perp(\mathcal{O}) &= \mathfrak{n}_{k-1}^\perp(\lambda), \quad \text{for any } \lambda \in \mathcal{O}. \end{aligned}$$

Since the restriction of  $\lambda \in \mathfrak{n}^*$  to the centre  $Z(\mathfrak{n})$  does not depend on the choice of the linear form  $\lambda \in \mathcal{O}$  and since  $[\mathfrak{n}, \mathfrak{n}_{k-1}] = \mathfrak{n}_k \subset Z(\mathfrak{n})$ , the restriction  $B_\lambda|_{\mathfrak{n} \times \mathfrak{n}_{k-1}}$  and the subspace  $\mathfrak{n}_{k-1}^\perp(\lambda)$  depend only on the coadjoint orbit  $\mathcal{O}$ . Consequently, the second definition in (16) is well-posed.

**Lemma 2.2.** *Let  $\mathcal{O}$  be a coadjoint orbit and  $\lambda \in \mathcal{O}$ . Then*

- (1)  $\mathfrak{r}_\lambda \subset \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ ;
- (2)  $\mathfrak{n}_2 \subset \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ ;
- (3)  $\mathfrak{n}_{k-1}^\perp(\mathcal{O})$  is a sub-algebra.

*Proof.* Condition (1) is immediate from the definitions. Conditions (2) and (3) follow from Jacobi's identity and the inclusion  $\mathfrak{n}_k \subset Z(\mathfrak{n})$ . In fact, if  $B_\lambda(T_1, \mathfrak{n}_{k-1}) = B_\lambda(T_2, \mathfrak{n}_{k-1}) = \{0\}$  then  $B_\lambda([T_1, T_2], \mathfrak{n}_{k-1}) = \lambda([ [T_1, T_2], \mathfrak{n}_{k-1} ]) \subset -\lambda([ [T_2, \mathfrak{n}_{k-1}], T_1 ]) - \lambda([ [\mathfrak{n}_{k-1}, T_1], T_2 ]) = \{0\}$ , since  $[T_j, \mathfrak{n}_{k-1}] \subset \mathfrak{n}_k \subset Z(\mathfrak{n})$ . □

**Lemma 2.3.** *Let  $\mathcal{O}$  be a coadjoint orbit and  $\lambda \in \mathcal{O}$ . The following properties are equivalent:*

- (1) *the restriction  $\lambda|_{\mathfrak{n}_k}$  is identically zero*
- (2)  $\mathfrak{n}_{k-1}^\perp(\mathcal{O}) = \mathfrak{n}$
- (3) *the projection of  $\mathfrak{n}_{k-1}^\perp(\mathcal{O})$  on  $\mathfrak{n}/\mathfrak{n}_2$  is surjective*

*Proof.* (1 $\Rightarrow$ 2) Since  $[\mathfrak{n}, \mathfrak{n}_{k-1}] = \mathfrak{n}_k$ , if the restriction  $\lambda|_{\mathfrak{n}_k}$  is identically zero, by definition, we have  $\mathfrak{n}_{k-1}^\perp(\lambda) = \mathfrak{n}$ .

(2 $\Rightarrow$ 1) From  $[\mathfrak{n}, \mathfrak{n}_{k-1}] = \mathfrak{n}_k$ , if  $\mathfrak{n}_{k-1}^\perp(\lambda) = \mathfrak{n}$ , we have  $\lambda|_{\mathfrak{n}_k} = 0$ .

(3 $\Leftrightarrow$ 2) Any subalgebra of  $\mathfrak{n}$  containing  $\mathfrak{n}_2$  and which projects onto  $\mathfrak{n}/\mathfrak{n}_2$  must coincide with the full algebra  $\mathfrak{n}$ .  $\square$

If the conditions of the above lemma are satisfied, any  $\pi \in \Pi_{\mathcal{O}}$  factors through a representation of  $N/\exp \mathfrak{n}_k$ , a  $(k-1)$ -step nilpotent group. Having in mind an induction process on the degree of nilpotency  $k$ , we shall assume that we are not in this case.

**Definition 2.4.** *If none of the conditions of the Lemma 2.3 are satisfied we will say that the coadjoint orbit  $\mathcal{O}$ , any linear form  $\lambda \in \mathcal{O}$  and any irreducible representation  $\pi \in \Pi_{\mathcal{O}}$  have maximal rank (equal to  $k$ ).*

Thus from now on we shall focus on coadjoint orbits  $\mathcal{O}$  of maximal rank.

**2.3. Adapted representations.** Let  $\mathcal{O}$  be any coadjoint orbit of maximal rank. In Section 3 we shall study the cohomological equation for a “generic” admissible  $X \in \mathfrak{n}$  restricted to an irreducible unitary representation  $\pi \in \Pi_{\mathcal{O}}$ . There we shall see, in fact, that the set of admissible  $X \in \mathfrak{n}$  equals  $\mathfrak{n} \setminus \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ .

The following lemma produces, for any coadjoint orbit  $\mathcal{O}$  of maximal rank and any  $X \in \mathfrak{n} \setminus \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ , a special irreducible unitary representation  $\pi \in \Pi_{\mathcal{O}}$  adapted to our goal of proving a priori estimates for the cohomological equation restricted to that representation.

**Lemma 2.5.** *Let  $X \in \mathfrak{n} \setminus \mathfrak{n}_{k-1}^\perp(\mathcal{O})$  and let  $Y \in \mathfrak{n}_{k-1}$  be any element such that  $B_\lambda(X, Y) \neq 0$  for all  $\lambda \in \mathcal{O}$ . There exists a codimension 1 ideal  $\mathfrak{n}' \subset \mathfrak{n}$  with  $X \notin \mathfrak{n}'$  and a unitary (irreducible) representation  $\pi \in \Pi_{\mathcal{O}}$  with the following properties:*

- (1) *the representation  $\pi$  is obtained inducing from  $N' := \exp \mathfrak{n}'$  to  $N$  a unitary irreducible representation  $\pi'$  of  $N'$  on a Hilbert space  $H'$ ;*
- (2) *the derived the representation  $\pi_*$  of the Lie algebra  $\mathfrak{n}$  satisfies*
  - (a)  $\pi_*(X) = \frac{\partial}{\partial t}$  on  $L^2(\mathbb{R}, H', dt)$ ,
  - (b)  $\pi_*(Y) = 2\pi i B_\lambda(X, Y) t \text{Id}_{H'}$  on  $L^2(\mathbb{R}, H', dt)$ ;
- (3) *there exists a constant  $C > 0$  such that*

$$(17) \quad |\langle X, U \rangle| \geq C^{-1} \frac{|B_\lambda(X, Y)|}{\|\lambda|_{\mathfrak{n}_k}\|}.$$

where  $U \in \mathfrak{n}$  denote the unit normal vector to  $\mathfrak{n}'$  with respect to an euclidean product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{n}$  fixed once for all.

*Proof.* Since, by Lemma 2.2 we have  $\mathfrak{n}_2 \subset \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ , for any  $\lambda \in \mathcal{O}$ , the subspace  $\mathfrak{n}' = \{T \in \mathfrak{n} \mid B_\lambda(T, Y) = 0\}$  is a codimension 1 ideal of  $\mathfrak{n}$  depending only on the coadjoint orbit  $\mathcal{O}$ . Let  $U \in \mathfrak{n}$  be a normal unit vector to  $\mathfrak{n}'$  with respect to an euclidean product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{n}$  fixed once for all. There is an orthogonal decomposition

$$(18) \quad X = \langle X, U \rangle U + W,$$

for some  $W \in \mathfrak{n}'$ . Since by definition of  $\mathfrak{n}'$  we have  $B_\lambda(W, Y) = 0$ , it follows immediately that

$$(19) \quad B_\lambda(X, Y) = \langle X, U \rangle B_\lambda(U, Y).$$

There exists a constant  $C > 0$  such that  $\|U\| = \|Y\| = 1$  implies that  $\|[U, Y]\| \leq C$  and  $Y \in \mathfrak{n}_{k-1}$  implies  $[U, Y] \in \mathfrak{n}_k$ . Hence

$$(20) \quad |B_\lambda(U, Y)| = |\lambda([U, Y])| \leq C \|\lambda|_{\mathfrak{n}_k}\|.$$

The lower bound (17) follows immediately from (19) and (20).

Let  $\mathfrak{m} \subset \mathfrak{n}'$  be a polarizing subalgebra for the restriction  $\lambda' = \lambda|_{\mathfrak{n}'}$  of  $\lambda$  to  $\mathfrak{n}'$ . Then  $\mathfrak{m}$  is also a polarizing subalgebra for  $\lambda$  on  $\mathfrak{n}$  ([CG90], Prop. 1.3.4), since, by definition, we have  $\mathfrak{r}_\lambda \subset \mathfrak{n}'$  for all  $\lambda \in \mathcal{O}$ .

Set

$$\Lambda = (\lambda, \mathfrak{m}), \quad \Lambda' = (\lambda', \mathfrak{m}), \quad \pi = \pi_\Lambda \quad \text{and} \quad \pi' = \pi_{\Lambda'}.$$

By induction in stages (see for example [CG90], §2.1), we have:

$$\pi = \text{Ind}_{\exp \mathfrak{m}}^N (\exp(\imath \lambda)) \approx \text{Ind}_{\exp \mathfrak{n}'}^N \left( \text{Ind}_{\exp \mathfrak{m}}^{\exp \mathfrak{n}'} (\exp(\imath \lambda')) \right) = \text{Ind}_{\exp \mathfrak{n}'}^N (\pi').$$

From (13)-(14), if we identify with  $\mathbb{R}$  the one-parameter subgroup generated by  $X$  and if we denote by  $H'$  the Hilbert space on which  $\pi_{\Lambda'}$  acts, up to a unitary equivalence, the representation  $\pi_\Lambda$  coincides with the representation of  $N$  on  $L^2(\mathbb{R}, H', dt)$  infinitesimally given by:

$$\pi_*(X)f(t) = \frac{\partial}{\partial t} f(t), \quad \pi_*(Y)f(t) = \pi'_*(\text{Ad}(\exp tX)Y)f(t),$$

for  $Y \in \mathfrak{n}'$  and all  $f$  in a dense subspace of  $L^2(\mathbb{R}, H', dt)$ , such as the subspace  $C_0^\infty(\mathbb{R}, H')$  of compactly supported  $C^\infty$  functions from  $\mathbb{R}$  to  $H'$ .

Since  $[X, Y] \in \mathfrak{n}_k \subset Z(\mathfrak{n})$  we have  $[X, [X, Y]] = 0$  and therefore

$$(21) \quad \text{Ad}(\exp tX)Y = Y + t[X, Y];$$

we also have  $\mathfrak{n}_k \subset \mathfrak{n}'$ , hence  $\mathfrak{n}_k \subset Z(\mathfrak{n}') \subset \mathfrak{m}'$  and  $\pi'|_{\mathfrak{n}_k} = \exp \imath \lambda$ ; from (15), we obtain that  $\pi_*(Y)$  is the operator pointwise given by:

$$\pi_*(Y) = t \pi'_*([X, Y]) + \pi'_*(Y) = 2\pi \imath t B_\lambda(X, Y) \text{Id}_{H'} + \pi'_*(Y).$$

We claim that there exist  $\lambda \in \mathcal{O}$  and a representation  $\pi' \in \Pi_{\lambda'}$  such that  $\pi'_*(Y) = 0$ . This will prove the lemma.

Since  $[X, Y] \in Z(\mathfrak{n})$ , the value  $B_\lambda(X, Y)$  does not depend on  $\lambda \in \mathcal{O}$ . Let  $\lambda \in \mathcal{O}$ . From (21) we have

$$(\text{Ad}^*(\exp t_0 X)\lambda)(Y) = \lambda(\text{Ad}(\exp(-t_0 X))Y) = \lambda(Y) - t_0 B_\lambda(X, Y)$$

which vanishes for  $t_0 = \lambda(Y)/B_\lambda(X, Y)$ . Thus we can and shall assume that we have chosen a linear form  $\lambda \in \mathcal{O}$  satisfying  $\lambda(Y) = 0$ .

Let  $\mathfrak{n}_k'' = \{T \in \mathfrak{n}_k \mid \lambda(T) = 0\}$ . By hypothesis  $\mathfrak{n}_k''$  is a central ideal in  $\mathfrak{n}$  of codimension 1 in  $\mathfrak{n}_k$ . We set

$$\bar{\mathfrak{n}}' = \mathfrak{n}'/\mathfrak{n}_k''$$

and we denote by  $\bar{\lambda}'$  the linear form induced on  $\bar{\mathfrak{n}}'_k$  by  $\lambda'$ . We also denote by

$$p' : N' = \exp \mathfrak{n}' \rightarrow N'/\exp \mathfrak{n}_k''$$

the canonical projection. We have  $\Pi_{\lambda'} = \{\bar{\pi}' \circ p' \mid \bar{\pi}' \in \Pi_{\bar{\lambda}'}\}$  since the central ideal  $\mathfrak{n}_k''$  is contained in every subalgebra  $\mathfrak{m} \subset \mathfrak{n}'$  polarizing for  $\lambda'$  and every representation  $\pi' \in \Pi_{\lambda'}$  is trivial on  $\exp \mathfrak{n}_k''$ . But now observe that the projection  $\bar{Y}$  of  $Y$  in  $\bar{\mathfrak{n}}'$  belongs to the centre  $Z(\bar{\mathfrak{n}}')$  of  $\bar{\mathfrak{n}}'$ . In fact, if  $T \in \mathfrak{n}'$ , we have  $[T, Y] \in \mathfrak{n}_k$  and then the condition  $B_\lambda(T, Y) = 0$  is equivalent to  $[T, Y] = 0 \pmod{\mathfrak{n}_k''}$ . We conclude that for every  $\bar{\pi}' \in \Pi_{\bar{\lambda}'}$  we have  $\bar{\pi}'_*(\bar{Y}) = \iota \bar{\lambda}'(\bar{Y}) = 0$  and consequently  $\pi'_*(Y) = 0$  for any  $\pi' \in \Pi_{\lambda'}$ .

This concludes the proof.  $\square$

### 3. THE COHOMOLOGICAL EQUATION IN REPRESENTATION

**3.1. Distributions and Sobolev spaces.** Let  $N$  be a connected, simply connected Lie group. Let  $\pi$  be a unitary representation of  $N$  on a Hilbert space  $H_\pi$ . Let  $\pi_*$  denote the derived representation of the Lie algebra  $\mathfrak{n}$  of  $N$  on  $H_\pi$ . The subspace  $C^\infty(H_\pi) \subset H_\pi$  is endowed with the Fréchet  $C^\infty$  topology, that is, the topology defined by the family of seminorms  $\{\|\cdot\|_{E_1, E_2, \dots, E_m} \mid m \in \mathbb{N} \text{ and } E_1, \dots, E_m \in \mathfrak{n}\}$  defined as follows:

$$\|v\|_{E_1, E_2, \dots, E_m} := \|\pi_*(E_1)\pi_*(E_2) \cdots \pi_*(E_m)v\|, \quad \text{for } v \in C^\infty(H_\pi).$$

**Definition 3.1.** Let  $\pi$  be a unitary representation of the Lie algebra  $\mathfrak{n}$  on the Hilbert space  $H_\pi$ . The space of Schwartz distributions for the representation  $\pi$  is defined as the dual space of  $C^\infty(H_\pi)$  (endowed with the Fréchet  $C^\infty$  topology). The space of Schwartz distributions for the representation  $\pi$  will be denoted  $\mathcal{D}'(H_\pi)$ .

The representation  $\pi_*$  extends in a canonical way to a representation (denoted by the same symbol) of the enveloping algebra  $\mathcal{U}(\mathfrak{n})$  of  $\mathfrak{n}$  on the Hilbert space  $H_\pi$ . Let  $\Delta \in \mathcal{U}(\mathfrak{n})$  a left-invariant second-order, positive elliptic operator on  $N$  fixed once for all. For example, the operator

$\Delta = -(V_1^2 + \dots + V_d^2)$ , where  $\{V_1, \dots, V_d\}$  is a basis of  $\mathfrak{n}$  as a vector space. For all  $\alpha \in \mathbb{R}^+$ , let  $W^\alpha(H_\pi) \subset H_\pi$  be the Sobolev space of vectors in the maximal domain of the essentially self-adjoint operator  $(I + \pi_*(\Delta))^{\alpha/2}$  endowed with the Hilbert space norm

$$\|v\|_\alpha := \|(I + \pi_*(\Delta))^{\alpha/2} v\|, \quad \text{for all } v \in W^\alpha(H_\pi).$$

Let also  $W^{-\alpha}(H_\pi) \subset \mathcal{D}'(H_\pi)$  be the dual Hilbert space of  $W^\alpha(H_\pi)$ . Then  $C^\infty(H_\pi)$  is the projective limit of the spaces  $W^\alpha(H_\pi)$  (and consequently  $\mathcal{D}'(H_\pi)$  is the inductive limit of  $W^{-\alpha}(H_\pi)$ ) as  $\alpha \rightarrow +\infty$ . A distribution  $D \in W^{-\alpha}(H_\pi)$  will be called a *distribution of order at most  $\alpha \in \mathbb{R}^+$* .

**Definition 3.2.** *Let  $\pi$  be a unitary representation of the Lie group  $N$  (with Lie algebra  $\mathfrak{n}$ ) on the Hilbert space  $H_\pi$ . The cohomological equation for  $X \in \mathfrak{n}$  in the representation  $\pi$  is the linear equation*

$$(22) \quad \pi_*(X)u = f \quad (\text{for the unknown vector } u \in H_\pi),$$

where  $f \in H_\pi$  is a given (sufficiently smooth) vector.

**Definition 3.3.** *For any  $X \in \mathfrak{n}$ , the space of  $X$ -invariant distributions for the representation  $\pi$  is defined as the space  $\mathcal{I}_X(H_\pi)$  of all distributional solutions  $D \in \mathcal{D}'(H_\pi)$  of the equation  $\pi_*(X)D = 0$ . Let*

$$\mathcal{I}_X^\alpha(H_\pi) := \mathcal{I}_X(H_\pi) \cap W^{-\alpha}(H_\pi)$$

be the subspace of invariant distributions of order at most  $\alpha \in \mathbb{R}^+$ .

It is immediate that all  $X$ -invariant distributions in  $\mathcal{D}'(H_\pi)$  yield obstructions to the existence of  $C^\infty$  solutions  $u \in C^\infty(H_\pi)$  of the cohomological equation and that  $X$ -invariant distributions of order at most  $\alpha \in \mathbb{R}^+$  yield obstructions to the existence of solutions  $u \in W^{\alpha+1}(H_\pi)$ .

Let  $\mathcal{O}$  be a codjoint orbit of maximal rank and let  $\pi \in \Pi_{\mathcal{O}}$  be an induced irreducible (adapted) unitary representation of  $N$  on  $H_\pi = L^2(\mathbb{R}, H')$  constructed as in Lemma 2.5. The space of  $C^\infty(H_\pi)$  of  $C^\infty$  vectors for the representation  $\pi$  can be characterized as follows.

Let  $\mathcal{S}(\mathbb{R}, C^\infty(H'))$  be the space of Schwartz functions on  $\mathbb{R}$  with values in the Fréchet space of  $C^\infty$  vectors for the representation  $\pi'$  on  $H'$  (cf. Lemma 2.5). The space  $\mathcal{S}(\mathbb{R}, C^\infty(H'))$  is by definition endowed with the Fréchet topology induced by the family of seminorms

$$\{\|\cdot\|_{i,j,E_1,E_2,\dots,E_m} \mid i,j,m \in \mathbb{N} \text{ and } E_1, \dots, E_m \in \mathfrak{n}'\}$$

defined as follows: for all  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ ,

$$(23) \quad \|f\|_{i,j,E_1,E_2,\dots,E_m} := \sup_{t \in \mathbb{R}} \|(1+t^2)^{j/2} \pi'_*(E_1) \cdots \pi'_*(E_m) f^{(i)}(t)\|_{H'}.$$

**Lemma 3.4.** *As topological vector spaces,*

$$C^\infty(H_\pi) \equiv \mathcal{S}(\mathbb{R}, C^\infty(H')) .$$

*Proof.* Let  $X, Y \in \mathfrak{n}$  and  $\mathfrak{n}' \subset \mathfrak{n}$  be as in Lemma 2.5. By formula (15), for any  $E \in \mathfrak{n}'$  and for all  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ ,

$$(24) \quad (\pi_*(E)f)(t) = \sum_{j=0}^k \frac{t^j}{j!} \pi'_*(\text{ad}(X)^j E) f(t).$$

We claim that the Schwartz space  $\mathcal{S}(\mathbb{R}, C^\infty(H')) \subset C^\infty(H_\pi)$  and that the embedding is continuous. In fact, for any  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ , we have

$$(25) \quad \|f\| \leq \sup_{t \in \mathbb{R}} \|(1+t^2)^{1/2} f(t)\|_{H'},$$

hence it follows from formula (24) that there exists a constant  $C_k > 0$  such that for any  $E \in \mathfrak{n}'$  and all  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ ,

$$(26) \quad \|\pi_*(E)f\| \leq C_k \max_{0 \leq j \leq k} \left[ \sup_{t \in \mathbb{R}} \|(1+t^2)^{\frac{1}{2}} t^j \pi'_*(\text{ad}(X)^j E) f(t)\|_{H'} \right].$$

In order to simplify the notation, let  $E^{(j)} := \text{ad}(X)^j E \in \mathfrak{n}'$  for any  $E \in \mathfrak{n}'$  and for any  $j \in \{0, \dots, k\}$ . By estimate (26) and by iterated applications of the identity (24), it follows that for any  $\ell \in \mathbb{N}$  there exists a constant  $C_{k,\ell} > 0$  such that, for any  $E_1, \dots, E_\ell \in \mathfrak{n}'$  and for all  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ ,

$$(27) \quad \|\pi_*(E_1) \cdots \pi_*(E_\ell) f\| \leq C_{k,\ell} \max_{0 \leq j_1, \dots, j_\ell \leq k} \left[ \sup_{t \in \mathbb{R}} \|(1+t^2)^{\frac{1}{2}} \prod_{\alpha=1}^{\ell} t^{j_\alpha} \pi'_*(E_\alpha^{(j_\alpha)}) f(t)\|_{H'} \right],$$

hence, for any  $E_1, \dots, E_\ell \in \mathfrak{n}'$ , any  $i \in \mathbb{N}$  and any  $f \in \mathcal{S}(\mathbb{R}, C^\infty(H'))$ ,

$$(28) \quad \|\pi_*(E_1) \cdots \pi_*(E_\ell) \pi_*(X)^i f\| \leq C_{k,\ell} \max_{0 \leq j_1, \dots, j_\ell \leq k} \left[ \sup_{t \in \mathbb{R}} \|(1+t^2)^{\frac{1}{2}} \prod_{\alpha=1}^{\ell} t^{j_\alpha} \pi'_*(E_\alpha^{(j_\alpha)}) f^{(i)}(t)\|_{H'} \right].$$

Since  $[X, \mathfrak{n}'] \subset \mathfrak{n}'$  the claim follows.

Conversely, we claim that  $C^\infty(H_\pi) \subset \mathcal{S}(\mathbb{R}, C^\infty(H'))$ . In fact, let  $f \in C^\infty(H_\pi)$  be a  $C^\infty$  vector. Since  $\pi_*(X)$  is the operator  $f \mapsto f'$ , we have  $f \in C^\infty(\mathbb{R}, H')$ . Since  $\pi_*(Y)$  equals, up to a non-zero (scalar) factor, the pointwise operator  $f(t) \mapsto t f(t)$ , we have  $(t \mapsto t^j f(t)) \in L^2(\mathbb{R}, H')$ , for all  $j \geq 0$ , hence  $f \in \mathcal{S}(\mathbb{R}, H')$ . Thus

$$(29) \quad C^\infty(H_\pi) \subset \mathcal{S}(\mathbb{R}, H')$$

and we therefore have a continuous linear operator

$$\mathcal{E}_t : C^\infty(H_\pi) \rightarrow H', \quad \mathcal{E}_t f = f(t).$$

For any  $E \in \mathfrak{n}'$  and any  $j \in \mathbb{N}$ , let  $T^j \otimes \pi'_*(E)$  be the densely defined linear operator on  $L^2(\mathbb{R}, H')$  uniquely determined by the identities

$$\mathcal{E}_t[T^j \otimes \pi'_*(E)]f := t^j \pi'_*(E) \mathcal{E}_t f, \quad \text{for all } t \in \mathbb{R}.$$

We will prove by (finite) induction that, for all  $\alpha \in \{0, \dots, k-1\}$ , if  $E \in \mathfrak{n}_{k-\alpha} \cap \mathfrak{n}'$ , then we have, for all  $j \in \mathbb{N}$  and for all  $f \in C^\infty(H_\pi)$ ,

$$(30) \quad \begin{aligned} f &\in \text{dom}[T^j \otimes \pi'_*(E)], \\ [T^j \otimes \pi'_*(E)]f &\in C^\infty(H_\pi). \end{aligned}$$

Remark that, since  $[T^j \otimes \pi'_*(E)]$  equals  $\pi_*(Y)^j[1 \otimes \pi'_*(E)]$  up to a non-zero scalar factor, the second line of formula (30) actually follows from

$$(31) \quad [1 \otimes \pi'_*(E)]f \in C^\infty(H_\pi).$$

If  $\alpha = 0$ , for any  $E_0 \in \mathfrak{n}_k \subset Z(\mathfrak{n})$  the operators  $\pi_*(E_0)$  and  $\pi'_*(E_0)$  are scalar multiples of the identity. Hence the claim (30) holds in this case.

Let us assume (30) for all  $E \in \mathfrak{n}_{k-\alpha}$  and for all  $j \in \mathbb{N}$ . Let  $E_{\alpha+1} \in \mathfrak{n}_{k-(\alpha+1)} \cap \mathfrak{n}'$ . Remark that the elements

$$E_\alpha^{(\beta)} := \text{ad}(X)^\beta E_{\alpha+1} \in \mathfrak{n}_{k-\alpha} \cap \mathfrak{n}', \quad \text{for all } \beta \in \{1, \dots, \alpha+1\}.$$

By formula (24) we have the identity,

$$(32) \quad \pi'_*(E_{\alpha+1}) \mathcal{E}_t f = \mathcal{E}_t \pi_*(E_{\alpha+1}) f - \sum_{\beta=1}^{\alpha+1} \frac{1}{\beta!} \mathcal{E}_t [T^\beta \otimes \pi'_*(E_\alpha^{(\beta)})] f.$$

where  $\pi_*(E_{\alpha+1})f \in C^\infty(H_\pi)$  and  $[T^i \otimes \pi'_*(E_\alpha^{(\beta)})]f \in C^\infty(H_\pi)$  by the induction hypothesis, since  $f \in C^\infty(H_\pi)$ . It follows by the inclusion (29) and the identity (32) that the claim (30) holds for all  $E_{\alpha+1} \in \mathfrak{n}_{k-\alpha} \cap \mathfrak{n}'$  and all  $j \in \mathbb{N}$ . The induction step is therefore completed.

It follows from (30) by another induction argument (on  $i \in \mathbb{N}$ ), that, for all  $E \in \mathfrak{n}'$  and all  $i, j \in \mathbb{N}$ , we have

$$(33) \quad \begin{aligned} f^{(i)} &\in \text{dom}[T^j \otimes \pi'_*(E)], \\ [T^j \otimes \pi'_*(E)]f^{(i)} &\in C^\infty(H_\pi). \end{aligned}$$

In fact, for  $i = 0$  the conditions (33) reduce to (30). Assume that (33) holds for  $i \in \mathbb{N}$  and for all  $j \in \mathbb{N}$ . Since  $[1 \otimes \pi'_*(E)]f^{(i)} \in C^\infty(H_\pi)$ , we have

$$[1 \otimes \pi'_*(E)]f^{(i+1)} = \pi_*(X)[1 \otimes \pi'_*(E)]f^{(i)} \in C^\infty(H_\pi) \subset \mathcal{S}(\mathbb{R}, H'),$$

hence  $f^{(i+1)} \in \text{dom}[1 \otimes \pi'_*(E)]$ . Next, for  $j \in \mathbb{N} \setminus \{0\}$ , we have the following immediate identity:

$$(34) \quad [T^j \otimes \pi'_*(E)]f^{(i+1)} = \pi_*(X)[T^j \otimes \pi'_*(E)]f^{(i)} - j[T^{j-1} \otimes \pi'_*(E)]f^{(i)},$$

which allows to conclude that (33) holds for  $i+1$ . The induction argument is therefore completed.

Finally, by successive applications of (33) we have that, for any  $i, j \in \mathbb{N}$ , any  $E_1, \dots, E_m \in \mathfrak{n}'$  and for all  $f \in C^\infty(H_\pi)$ ,

$$(35) \quad \begin{aligned} f^{(i)} &\in \text{dom}[T^j \otimes \pi'_*(E_1) \cdots \pi'_*(E_m)], \\ [T^j \otimes \pi'_*(E_1) \cdots \pi'_*(E_m)]f^{(i)} &\in C^\infty(H_\pi). \end{aligned}$$

In conclusion, we have proved that  $\mathcal{S}(\mathbb{R}, C^\infty(H')) \subset C^\infty(H_\pi)$  and that the inclusion is continuous and surjective, hence it is an isomorphism of Fréchet spaces by the open mapping theorem (or by a direct argument based on the proof of surjectivity given above).  $\square$

**3.2. Main Estimates.** In this section we describe all invariant distributions and we prove Sobolev bounds for the Green operator of the cohomological equation in each irreducible unitary representation of maximal rank (determined by a coadjoint orbit  $\mathcal{O}$  of maximal rank) for any admissible vector  $X \in \mathfrak{n} \setminus \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ .

Let  $\langle \cdot, \cdot \rangle$  be an euclidean product on the Lie algebra  $\mathfrak{n}$  fixed once for all and let  $\|\cdot\|$  denote the corresponding norm.

Let  $\mathcal{O}$  be any coadjoint orbit. Since the restrictions  $\lambda|_{\mathfrak{n}_k}$  and  $B_\lambda|_{\mathfrak{n} \times \mathfrak{n}_{k-1}}$  do not depend on the choice of the linear form  $\lambda \in \mathcal{O}$ , the following definitions are well-posed.

Let

$$(36) \quad \begin{aligned} w_k(\mathcal{O}) &:= \|\lambda|_{\mathfrak{n}_k}\| = \max_{Z \in \mathfrak{n}_k \setminus \{0\}} \frac{|\lambda(Z)|}{\|Z\|}; \\ w_Z(\mathcal{O}) &:= \|\lambda|_{Z(\mathfrak{n})}\| = \max_{Z \in Z(\mathfrak{n}) \setminus \{0\}} \frac{|\lambda(Z)|}{\|Z\|}. \end{aligned}$$

Since  $\mathfrak{n}_k \subset Z(\mathfrak{n})$  we have  $w_k(\mathcal{O}) \leq w_Z(\mathcal{O})$ .

For all  $(X, Y) \in \mathfrak{n} \times \mathfrak{n}_{k-1}$ , let

$$(37) \quad \begin{aligned} \delta_{\mathcal{O}}(X, Y) &:= |B_\lambda(X, Y)|, \text{ for any } \lambda \in \mathcal{O}, \text{ and} \\ \delta_{\mathcal{O}}(X) &:= \max\{\delta_{\mathcal{O}}(X, Y) \mid Y \in \mathfrak{n}_{k-1} \text{ and } \|Y\| = 1\}. \end{aligned}$$

By definition we have  $\delta_{\mathcal{O}}(X) > 0$  if and only if  $X \notin \mathfrak{n}_{k-1}^\perp(\mathcal{O})$ . The latter condition is non-empty if and only if  $\mathcal{O}$  has maximal rank, in which case it holds except for a subspace of positive codimension.

**Lemma 3.5.** *Let  $\delta_{\mathcal{O}}(X) > 0$ , and  $\pi \in \Pi_{\mathcal{O}}$  an irreducible essentially skew-adjoint representation of  $\mathfrak{n}$  on a Hilbert space  $H_{\pi}$ . Let  $\mathcal{I}_X(H_{\pi}) \subset \mathcal{D}'(H_{\pi})$  be the space of  $X$ -invariant distributions, i.e. of linear functionals which vanish on  $\text{Ran}(X|C^{\infty}(H_{\pi})) \subset C^{\infty}(H_{\pi})$ . The space  $\mathcal{I}_X(H_{\pi})$  has a countable basis  $\mathcal{B}_X(H_{\pi})$  with the following properties:*

- (1) *each  $D \in \mathcal{B}_X(H_{\pi})$  has Sobolev order equal to  $1/2$ , in the sense that  $D \in W^{-\alpha}(H_{\pi})$  for any  $\alpha > 1/2$ ;*
- (2) *for each  $D \in \mathcal{B}_X(H_{\pi})$  and each  $\alpha > 1/2$ , there exists a constant  $C := C(D, \alpha) > 0$  such that the following holds: for any  $Y \in \mathfrak{n}_{k-1}$  with  $\delta_{\mathcal{O}}(X, Y) > 0$  and for all  $f \in W^{\alpha}(H_{\pi})$ , we have*

$$(38) \quad |D(f)| \leq C \delta_{\mathcal{O}}(X, Y)^{-1/2} \|(I - \pi_*(Y)^2)^{\alpha/2} f\|.$$

*Proof.* Let  $\mathcal{O} \subset \mathfrak{n}^*$  be a coadjoint orbit such that  $\delta_{\mathcal{O}}(X) > 0$ , let  $Y \in \mathfrak{n}_{k-1}$  be such that  $\delta_{\mathcal{O}}(X, Y) > 0$  and let  $\pi \in \Pi_{\mathcal{O}}$ . Since the statement of the theorem depends only on the unitary equivalence class of the representation, we can assume by Lemma 2.5 that  $\pi$  is obtained from an irreducible representation  $\pi'$  of an ideal  $\mathfrak{n}' \subset \mathfrak{n}$  of codimension 1 on a Hilbert space  $H'$  by adjoining  $X \in \mathfrak{n} \setminus \mathfrak{n}'$ , hence  $\pi_*(X)$  is the derivative operator  $\frac{\partial}{\partial t}$  on  $H_{\pi} \equiv L^2(\mathbb{R}, H', dt)$ , and that  $\pi_*(Y) = 2\pi i t \delta_{\mathcal{O}}(X, Y) \text{Id}_{H'}$ .

Let  $\mathcal{B}' \subset H'$  be a basis of the Hilbert space  $H'$  and let  $\mathcal{B}_X(H_{\pi})$  be the set of linear functionals  $D_e$ ,  $e \in \mathcal{B}'$ , defined as follows. For any  $e \in \mathcal{B}'$ , for all  $f \in \mathcal{S}(\mathbb{R}, H')$ , let

$$(39) \quad D_e(f) := \int_{\mathbb{R}} \langle f(t), e \rangle_{H'} dt.$$

By Hölder inequality, for all  $\alpha > 1/2$  there exists a constant  $C_{\alpha} > 0$  such that the following (a priori) estimate holds:

$$(40) \quad |D_e(f)| \leq \frac{C_{\alpha}}{\delta_{\mathcal{O}}(X, Y)^{1/2}} |e|_{H'} \|(I - \pi_*(Y)^2)^{\alpha/2} f\|;$$

hence  $D_e \in W^{-\alpha}(H)$  for all  $\alpha > 1/2$ ; in particular  $D_e \in \mathcal{D}'(H_{\pi})$ .

Clearly  $D_e$  is invariant by translations, hence  $D_e \in \mathcal{I}_X(H_{\pi})$ . It remains to be proved only that  $\mathcal{B}_X(H_{\pi})$  is a basis of  $\mathcal{I}_X(H_{\pi})$ . Since by Lemma 3.4 we have  $C^{\infty}(H_{\pi}) \equiv \mathcal{S}(\mathbb{R}, C^{\infty}(H'))$ , if  $D_e(f) = 0$  for all  $e \in \mathcal{B}'$ , then  $\int_{\mathbb{R}} f(s) ds = 0 \in C^{\infty}(H')$  and the function

$$G(t) := \int_{-\infty}^t f(s) ds \in \mathcal{S}(\mathbb{R}, C^{\infty}(H')).$$

It follows that  $D(f) = 0$  for all  $D \in \mathcal{I}_X(H_{\pi})$ , since there exists  $g \in C^{\infty}(H_{\pi})$  such that  $f = \pi(X)g$ . Hence, by the Hahn-Banach theorem,  $\mathcal{B}_X(H_{\pi})$  is a (countable) system of generators for the (closed) subspace  $\mathcal{I}_X(H_{\pi}) \subset \mathcal{D}'(H_{\pi})$ . It is immediate to verify that, since  $\mathcal{B}' \subset H'$  is linearly

independent, the system  $\mathcal{B}_X(H_\pi)$  is also linearly independent, hence it is a basis of  $\mathcal{I}_X(H_\pi)$ .  $\square$

**Theorem 3.6.** *Let  $\delta_{\mathcal{O}}(X) > 0$ , and let  $\pi \in \Pi_{\mathcal{O}}$  be an irreducible essentially skew-adjoint representation of  $\mathfrak{n}$  on a Hilbert space  $H_\pi$ .*

- (1) *There exists a Green operator  $G_X : W^\alpha(H_\pi) \rightarrow W^\beta(H_\pi)$ , defined for all  $\alpha > 1/2$  and  $\beta < -1/2$ , such that, for all  $f \in W^\alpha(H_\pi)$ , the distribution  $u := G_X f$  is a solution of the cohomological equation  $\pi_*(X)u = f$ ; there exists a constant  $C := C(X, \alpha) > 0$  such that*

$$\|G_X f\|_\beta \leq C \delta_{\mathcal{O}}(X)^{-1} \|f\|_\alpha.$$

- (2) *If  $f \in W^\alpha(H_\pi)$ ,  $\alpha > 1$ , and  $D(f) = 0$ , for all  $D \in \mathcal{I}_X(H_\pi)$ , then  $G_X f \in W^\beta(H_\pi)$ , for all  $\beta < (\alpha - 1)/k$ ; furthermore there exists a constant  $C' := C(X, \alpha, \beta) > 0$  such that*

$$\|G_X f\|_\beta \leq C' \max\{1, w_k(\mathcal{O})^\beta\} \max\{1, \delta_{\mathcal{O}}(X)^{-1-k\beta}\} \|f\|_\alpha.$$

*Proof.* Let  $\mathcal{O} \subset \mathfrak{n}^*$  be a coadjoint orbit and let  $\pi \in \Pi_{\mathcal{O}}$ . If  $\delta_{\mathcal{O}} := \delta_{\mathcal{O}}(X) > 0$ , by Lemma 2.5 we can assume that the representation  $\pi$  is obtained from an irreducible representation  $\pi'$  of a codimension 1 ideal  $\mathfrak{n}'$  on a Hilbert space  $H'$  by adjoining  $X \in \mathfrak{n} \setminus \mathfrak{n}'$ , hence  $\pi_*(X)$  is the derivative operator  $\frac{\partial}{\partial t}$  on  $H_\pi \equiv L^2(\mathbb{R}, H', dt)$ , and that there exists  $Y \in \mathfrak{n}_{k-1}$  such that  $\pi_*(Y) = 2\pi i t \delta_{\mathcal{O}} \text{Id}_{H'}$ . In fact, by compactness of the unit sphere in  $\mathfrak{n}_{k-1}$ , there exists  $Y \in \mathfrak{n}_{k-1}$  such that  $\|Y\| = 1$  and  $\delta_{\mathcal{O}}(X, Y) = \delta_{\mathcal{O}} > 0$ .

The operator  $G_X : C_0(\mathbb{R}, H') \rightarrow B(\mathbb{R}, H')$  defined for  $f \in C_0(\mathbb{R}, H')$  by

$$(41) \quad G_X f(t) := \int_{-\infty}^t f(s) ds, \quad \text{for all } t \in \mathbb{R},$$

admits a bounded extension  $G_X : W^\alpha(H_\pi) \rightarrow C_B(\mathbb{R}, H')$ , for all  $\alpha > 1/2$ ; in fact, if  $f \in W^\alpha(H_\pi)$ ,  $\alpha > 1/2$ , by Hölder inequality there exists a constant  $C_\alpha > 0$  such that

$$(42) \quad \int_{-\infty}^t |f(s)|_{H'} ds \leq \frac{C_\alpha}{\delta_{\mathcal{O}}^{1/2}} \|(I - \pi(Y)^2)^{\alpha/2} f\|.$$

A similar estimate shows that  $C_B(\mathbb{R}, H') \subset W^{-\alpha}(H_\pi)$  for all  $\alpha > 1/2$ . In fact, if  $u \in C_B(\mathbb{R}, H')$ , for all  $\alpha > 1/2$  there exists  $C_\alpha > 0$  such that, if  $v \in W^\alpha(H_\pi)$ ,

$$(43) \quad |(u, v)| \leq \int_{\mathbb{R}} |\langle u(s), v(s) \rangle_{H'}| ds \leq \frac{C_\alpha}{\delta_{\mathcal{O}}^{1/2}} \|u\|_\infty \|v\|_\alpha.$$

It follows that  $G_X : W^\alpha(H_\pi) \rightarrow W^\beta(H_\pi)$  is well-defined and bounded for all  $\alpha > 1/2$  and all  $\beta < -1/2$ , and there exists  $C_{\alpha,\beta} > 0$  such that, for all  $f \in W^\alpha(H_\pi)$ ,

$$(44) \quad \|G_X f\|_\beta \leq \frac{C_{\alpha,\beta}}{\delta_{\mathcal{O}}} \|f\|_\alpha$$

By construction  $\pi_*(X)G_X f = f$  in  $W^{\beta-1}(H_\pi)$ , for all  $f \in W^\alpha(H_\pi)$ ,  $\alpha > 1/2$  and  $\beta < -1/2$ , and  $G_X \pi_*(X)u = u$  in  $W^\beta(H_\pi)$ , for all  $u \in W^{\alpha+1}(H_\pi)$ ,  $\alpha > 1/2$  and  $\beta < -1/2$ .

Finally, if  $f \in W^\alpha(H_\pi)$ , for  $\alpha > 1/2$ , and  $D(f) = 0$ , for all  $D \in \mathcal{I}_X^\alpha(H_\pi)$ ,

$$(45) \quad G_X f(t) := \int_{-\infty}^t f(s) ds = - \int_t^{+\infty} f(s) ds, \quad \text{for all } t \in \mathbb{R}.$$

It follows that

$$(46) \quad \|\pi_*(Y)^\ell G_X f\|^2 \leq \int_0^{+\infty} \left( |2\pi\delta_{\mathcal{O}} t|^\ell \int_t^{+\infty} |f(s)|_{H'} ds \right)^2 dt \\ + \int_{-\infty}^0 \left( |2\pi\delta_{\mathcal{O}} t|^\ell \int_{-\infty}^t |f(s)|_{H'} ds \right)^2 dt.$$

Since, for all  $\alpha > 1$ ,

$$(47) \quad C_{\alpha,\ell}^2 := \int_0^{+\infty} (2\pi t)^{2\ell} \left( \int_t^{+\infty} (1 + 4\pi s^2)^{-(\ell+\alpha)} ds \right) dt \\ + \int_{-\infty}^0 (2\pi t)^{2\ell} \left( \int_{-\infty}^t (1 + 4\pi s^2)^{-(\ell+\alpha)} ds \right) dt < +\infty,$$

by Hölder inequality and change of variables,

$$(48) \quad \|\pi_*(Y)^\ell G_X f\| \leq \frac{C_{\alpha,\ell}}{\delta_{\mathcal{O}}} \|(I - \pi_*(Y)^2)^{\frac{\ell+\alpha}{2}} f\|.$$

Let  $\mathfrak{U}(\mathfrak{n})$  and  $\mathfrak{U}(\mathfrak{n}')$  be the enveloping algebras of  $\mathfrak{n}$  and  $\mathfrak{n}'$  respectively. Clearly from the inclusion  $\mathfrak{n}' \subset \mathfrak{n}$  we have that  $\mathfrak{U}(\mathfrak{n}')$  is contained in  $\mathfrak{U}(\mathfrak{n})$  as a subalgebra. In addition, since  $\mathfrak{n}'$  is an ideal of  $\mathfrak{n}$ ,  $(\text{Ad}(\exp tX))_{t \in \mathbb{R}}$  is a one-parameter group of automorphisms of  $\mathfrak{n}'$  which extends to a one-parameter group  $(\exp t\delta)_{t \in \mathbb{R}}$  of automorphisms of  $\mathfrak{U}(\mathfrak{n}')$ . The generator  $\delta$  of this group of automorphisms is the derivation on  $\mathfrak{U}(\mathfrak{n}')$  obtained by extending the derivation  $\text{ad}(X)$  of  $\mathfrak{n}'$  from  $\mathfrak{n}'$  to  $\mathfrak{U}(\mathfrak{n}')$ . Observe that from the nilpotency of  $\mathfrak{n}$  it follows that, for any  $P \in \mathfrak{U}(\mathfrak{n}')$  there exists a first integer  $[P]$  such that  $\delta^{[P]+1}P = 0$ .

We recall that by Lemma 3.4 we have  $C^\infty(H_\pi) = \mathcal{S}(\mathbb{R}, C^\infty(H'))$ . Let  $\mathcal{E}_t : C^\infty(H_\pi) \rightarrow C^\infty(H')$  be the linear (evaluation) operator defined by  $\mathcal{E}_t f = f(t)$ . By definition of the representation  $\pi_*$  on  $C^\infty(H_\pi)$  (cf. formula (15)), we have, for any  $P \in \mathfrak{U}(\mathfrak{n}')$  and any  $E \in \mathfrak{n}'$ ,

$$(49) \quad \mathcal{E}_t \pi_*(P) = \pi'_*(e^{t\delta} P) \mathcal{E}_t = \sum_{j=0}^{[P]} \frac{t^j}{j!} \pi'_*(\delta^j P) \mathcal{E}_t, \quad \text{for all } t \in \mathbb{R},$$

from which we obtain, for all  $s, t \in \mathbb{R}$ ,

$$(50) \quad \pi'_*(e^{(t-s)\delta} P) \mathcal{E}_t = \mathcal{E}_t \pi_*(e^{-s\delta} P) = \sum_{j=0}^{[P]} \frac{(-s)^j}{j!} \mathcal{E}_t \pi_*(\delta^j P).$$

Since the action of  $\pi_*(Y)$  on  $C^\infty(H_\pi)$  can be rewritten as

$$(51) \quad \mathcal{E}_t \pi_*(Y^j) = (2\pi i \delta_{\mathcal{O}} t)^j \mathcal{E}_t, \quad \text{for all } j \in \mathbb{N} \text{ and } t \in \mathbb{R},$$

setting  $t = s$  in the (50) we obtain, for all  $t \in \mathbb{R}$ ,

$$(52) \quad \pi'_*(P) \mathcal{E}_t = \mathcal{E}_t \pi(e^{-t\delta} P) = \sum_{j=0}^{[P]} \frac{(-1)^j}{j! (2\pi i \delta_{\mathcal{O}})^j} \mathcal{E}_t \pi_*(Y^j \delta^j P).$$

Since  $\mathcal{E}_t G_X = \int_{-\infty}^t \mathcal{E}_s ds$  we have, for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} \mathcal{E}_t \pi_*(P) G_X &= \pi'_*(e^{t\delta} P) \mathcal{E}_t G_X = \pi'_*(e^{t\delta} P) \int_{-\infty}^t \mathcal{E}_s ds \\ &= \sum_{j=0}^{[P]} \frac{t^j}{j!} \pi'_*(\delta^j P) \int_{-\infty}^t \mathcal{E}_s ds = \sum_{j=0}^{[P]} \frac{t^j}{j!} \int_{-\infty}^t \pi'_*(\delta^j P) \mathcal{E}_s ds \\ &= \sum_{j=0}^{[P]} \frac{t^j}{j!} \sum_{\ell=0}^{[P]} \frac{(-1)^\ell}{\ell! (2\pi i \delta_{\mathcal{O}})^\ell} \int_{-\infty}^t \mathcal{E}_s \pi_*(Y^\ell \delta^{j+\ell} P) ds \\ &= \sum_{j=0}^{[P]} \sum_{\ell=0}^{[P]} \frac{t^j}{j!} \frac{(-1)^\ell}{\ell! (2\pi i \delta_{\mathcal{O}})^\ell} \mathcal{E}_t G_X \pi_*(Y^\ell \delta^{j+\ell} P) \\ &= \mathcal{E}_t \sum_{j=0}^{[P]} \sum_{\ell=0}^{[P]} \frac{(-1)^\ell}{j! \ell! (2\pi i \delta_{\mathcal{O}})^{j+\ell}} \pi_*(Y^j) G_X \pi_*(Y^\ell \delta^{j+\ell} P) \\ &= \mathcal{E}_t \sum_{m=0}^{[P]} \frac{1}{(2\pi i \delta_{\mathcal{O}})^m} \sum_{\ell+j=m} \frac{(-1)^\ell}{j! \ell!} \pi_*(Y^j) G_X \pi_*(Y^\ell \delta^m P), \end{aligned}$$

hence, using (48), for all  $\alpha > 1$ , there exists a constant  $C_{\alpha,[P]} > 0$  such that for all  $f \in C^\infty(H_\pi)$ , we have

$$(53) \quad \|\pi_*(P)G_X f\| \leq \sum_{m=0}^{[P]} \frac{1}{(2\pi\delta_{\mathcal{O}})^m} \sum_{\ell+j=m} \frac{1}{j!\ell!} \frac{C_{\alpha,j}}{\delta_{\mathcal{O}}} \|(I - \pi(Y)^2)^{\frac{j+\alpha}{2}} \pi_*(Y^\ell \delta^m P)f\| \leq C_{\alpha,[P]} \sum_{m=0}^{[P]} \delta_{\mathcal{O}}^{-m-1} \|(I - \pi(Y)^2)^{\frac{m+\alpha}{2}} \pi_*(\delta^m P)f\|.$$

Let  $\Delta$  the positive definite Laplacian associated to an euclidean product  $\langle \cdot, \cdot \rangle$  on the Lie algebra  $\mathfrak{n}$ . Then for any orthonormal basis  $\mathcal{B}$  of  $\mathfrak{n}$  we have  $\Delta = -\sum_{V \in \mathcal{B}} V^2$ . Let  $U \in \mathfrak{n}$  be a normal unit vector to  $\mathfrak{n}'$ . Without loss of generality we can assume that  $U \in \mathcal{B}$ . We set  $\Delta_0 = -\sum_{V \in \mathcal{B} \setminus \{U\}} V^2$ ; we have  $\Delta_0 \in \mathfrak{U}(\mathfrak{n}')$  and  $\Delta = -U^2 + \Delta_0$ . Clearly  $[\Delta_0] \leq 2(k-1)$  and  $[\Delta_0^r] \leq 2(k-1)r$ . It follows by (53) that there exists a constant  $C := C(k, r, \alpha) > 0$  such that

$$(54) \quad \|\pi_*(\Delta_0^r)G_X f\| \leq C \max\{1, \delta_{\mathcal{O}}^{-1-2(k-1)r}\} \cdot \sum_{m=0}^{2(k-1)r} \|(I - \pi(\Delta_0))^{\frac{m+\alpha}{2}} \pi_*(\delta^m \Delta_0^r)f\|.$$

Since  $U$  is a unit vector orthogonal to  $\mathfrak{n}'$ , there exists  $W \in \mathfrak{n}'$  such that

$$(55) \quad U = \langle X, U \rangle^{-1} (X - W).$$

Since  $\mathfrak{n}' \subset \mathfrak{n}$  is an ideal and  $\pi(X)G_X f = f$ , it follows by estimate (17) in Lemma 2.5 that there exists a constant  $C' := C'(k, r, \alpha, X) > 0$  such that

$$(56) \quad \begin{aligned} \|\pi_*(U)^{2r}G_X f\| &\leq \left(\frac{w_k(\mathcal{O})}{\delta_{\mathcal{O}}}\right)^{2r} \|\pi_*(X - W)^{2r}G_X f\| \\ &\leq C' \left(\frac{w_k(\mathcal{O})}{\delta_{\mathcal{O}}}\right)^{2r} (\|f\|_{2r-1} + \|\pi_*(\Delta_0)^r G_X f\|). \end{aligned}$$

By bounds (54) and (56), there exists  $C''' := C'''(k, r, \alpha, X) > 0$  such that

$$(57) \quad \|G_X f\|_{2r} \leq C''' \max\{1, w_k(\mathcal{O})^{2r}\} \max\{1, \delta_{\mathcal{O}}^{-1-2rk}\} \|f\|_{2kr+\alpha}.$$

Finally by interpolation for all  $\alpha > \beta > 0$  such that  $\alpha > 1 + k\beta$  there exists a constant  $C_{\alpha,\beta} := C_{\alpha,\beta}(k, X) > 0$  such that

$$(58) \quad \|G_X f\|_{\beta} \leq C_{\alpha,\beta} \max\{1, w_k(\mathcal{O})^{\beta}\} \max\{1, \delta_{\mathcal{O}}^{-1-k\beta}\} \|f\|_{\alpha}.$$

□

## 4. THE COHOMOLOGICAL EQUATION FOR NILFLOWS

**4.1. The Howe-Richardson multiplicity formula.** Let  $N$  be a  $k$ -step nilpotent Lie group with a minimal set of generators  $\{E_1, \dots, E_n\} \subset \mathfrak{n}_1$ . Let  $\Gamma$  be a lattice in  $N$ . The nilmanifold  $\mathcal{N} = \Gamma \backslash N$  carries a unique  $N$ -invariant probability measure  $\mu$ , locally given by the Haar measure of  $N$ ; for each  $X \in \mathfrak{n}$  we denote by  $(\phi_X^t)_{t \in \mathbb{R}}$  the flow on  $\mathcal{N}$  given by the right action of the one-parametre subgroup  $\{\exp tX \mid t \in \mathbb{R}\}$ .

The Hilbert space  $L^2(\mathcal{N}, \mu)$  decomposes under the right action of  $N$  into a countable Hilbert sum  $\bigoplus_{i \in \mathbb{N}} H_i$  of irreducible closed subspaces  $H_i$ .

The Howe-Richardson multiplicity formula [How71, Ric71] tells us which irreducible representations from the unitary dual  $\hat{N}$  of  $N$  appear in the decomposition  $\bigoplus_{i \in \mathbb{N}} H_i$ . To state the formula, recall that a couple  $(\chi, M)$  is called a *maximal character* if there is  $\lambda \in \mathfrak{n}^*$  and a polarizing subalgebra  $\mathfrak{m} \subset \mathfrak{n}$  for  $\lambda$  such that:

- (1)  $M = \exp \mathfrak{m}$ ;
- (2)  $\chi$  is the one-dimensional representation of  $M$  defined by

$$\exp W \in M \mapsto e^{2\pi i \lambda(W)};$$

a maximal character  $(\chi, M)$  is called *maximal for  $\Gamma$*  if, in addition, the following conditions are satisfied:

- (3)  $M$  intersects  $\Gamma$  into a lattice, i.e.  $M \cap \Gamma \backslash M$  is compact;
- (4)  $\chi$  is trivial on  $M \cap \Gamma$ :  $\chi(M \cap \Gamma) = \{1\}$ .

As we recalled in §2.1, when  $(\chi, M)$  ranges over the set of maximal characters, the family of representations  $\text{Ind}_M(\chi)$  exhaust the unitary dual  $\hat{N}$ . The multiplicity formula states that  $\text{Ind}_M(\chi)$  appears in  $L^2(\mathcal{N}, \mu)$  iff  $(\chi, M)$  is a maximal integral character for  $\Gamma$ , with multiplicity given by the cardinal of closed orbits  $xM$  of  $M$  on  $\mathcal{N}$  such that  $\chi(\text{Stab}_M(x)) = \{1\}$ . This discussion motivates the following:

**Definition 4.1.** A linear form  $\lambda \in \mathfrak{n}^*$  is called *integral* (with respect to  $\Gamma$ ) if there exists a polarizing subalgebra  $\mathfrak{m} \subset \mathfrak{n}$  such that the pair  $(\exp(2\pi i \lambda), \exp \mathfrak{m})$  is a maximal integral character for  $\Gamma$ . A coadjoint orbit  $\mathcal{O} \subset \mathfrak{n}$  will be called *integral* (with respect to  $\Gamma$ ) if some  $\lambda \in \mathcal{O}$  is integral (hence all  $\lambda \in \mathcal{O}$  are).

**Definition 4.2.** Let  $\Gamma$  be a lattice in  $N$ . A linear form  $\lambda \in \mathfrak{n}^*$  is called *weakly integral* (with respect to  $\Gamma$ ) if  $\lambda(E) \in \mathbb{Z}$  for all  $E \in \log Z(\Gamma)$ . A coadjoint orbit  $\mathcal{O} \subset \mathfrak{n}$  will be called *weakly integral* (with respect to  $\Gamma$ ) if some  $\lambda \in \mathcal{O}$  is weakly integral (hence all  $\lambda \in \mathcal{O}$  are).

Since  $Z(\Gamma) = \Gamma \cap Z(N) \subset \exp \mathfrak{m}$  for any maximal polarizing subalgebra  $\mathfrak{m} \subset \mathfrak{n}$ , any integral linear form (coadjoint orbit) is also weakly integral.

In fact, since the pair  $(\exp(2\pi i\lambda), \exp \mathfrak{m})$  is a maximal integral character,  $\exp(2\pi i\lambda(W)) = 1$ , hence  $\lambda(W) \in \mathbb{Z}$ , for all  $W \in \log Z(\Gamma)$ .

It follows from the above discussion that all irreducible unitary subrepresentations occurring in  $L^2(\mathcal{N}, \mu)$  correspond are induced by Kirillov's by integral coadjoint orbits.

**4.2. Diophantine elements.** Let  $A$  the abelian group of rank  $n$  given by  $A = N/(N, N)$  and let  $\bar{\Gamma} = \Gamma/(\Gamma, \Gamma)$ ; then  $\bar{\Gamma}$  is a lattice in  $A$ . Let  $\{\exp \bar{E}_1, \dots, \exp \bar{E}_n\}$  denote a set of generators of  $\bar{\Gamma}$ , with  $\bar{E}_i \in \text{Lie}(A) \approx \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ . It is plain that the  $\bar{E}_i$ 's form a basis of  $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ .

We shall say that an element  $X \in \mathfrak{n}$  is *Diophantine (with respect to  $\Gamma$ ) of exponent  $\tau \geq 0$*  if the projection  $\bar{X}$  of  $X$  in  $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$  is Diophantine of exponent  $\tau \geq 0$  for the lattice  $\bar{\Gamma}$  in the standard sense: that is, if we let  $\bar{X} = \omega_1(X)\bar{E}_1 + \dots + \omega_n(X)\bar{E}_n$ , there exists a constant  $K > 0$  such that, for all  $M = (m_1, \dots, m_n) \in \mathbb{Z}^n \setminus \{0\}$ ,

$$(59) \quad |\langle M, \Omega_X \rangle| := \left| \sum_{i=1}^n m_i \omega_i(X) \right| \geq \frac{K}{|M|^{n-1+\tau}}.$$

The set of Diophantine elements  $X \in \mathfrak{n}$  of exponent  $\tau \geq 0$  will be denoted by  $\text{DC}_\tau(\mathfrak{n})$ . The subset of all Diophantine elements  $\text{DC}(\mathfrak{n}) := \cup_\tau \text{DC}_\tau(\mathfrak{n}) \subset \mathfrak{n}$  has full measure.

Let  $E_1^1, E_2^1, \dots, E_{n_1}^1, E_1^2, \dots, E_{n_2}^2, \dots, E_1^k, \dots, E_{n_k}^k$ , (with  $n_1 = n$ ), a Malcev basis for  $\mathfrak{n}$  through the descending central series  $\mathfrak{n}_j$  and strongly based at  $\Gamma$ , that is a basis of  $\mathfrak{n}$  satisfying the following properties:

- (1) if we drop the first  $\ell$  elements of the basis we obtain a basis if a subalgebra of codimension  $\ell$  of  $\mathfrak{n}$ .
- (2) if we set  $\mathcal{E}^j := \{E_1^j, \dots, E_{n_j}^j\}$  the elements of the set  $\mathcal{E}^j \cup \mathcal{E}^{j+1} \cup \dots \cup \mathcal{E}^k$  form a basis of  $\mathfrak{n}_j$
- (3) every element of  $\Gamma$  can be written as a product

$$\exp m_1^1 E_1^1 \dots \exp m_{n_1}^1 E_{n_1}^1 \dots \exp m_1^k E_1^k \dots \exp m_{n_k}^k E_{n_k}^k$$

with integral coefficients  $m_i^j$ .

The existence of such a basis is obtained combining the proof of Theorems 1.1.13 and 5.1.6 of [CG90].

It is also clear that we can assume that the basis  $\{\bar{E}_1, \dots, \bar{E}_n\}$  of  $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ , appearing above, is given by the projections in  $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$  of  $E_1^1, E_2^1, \dots, E_n^1$ .

We define an Euclidean product on  $\mathfrak{n}$  by making the basis  $E_j^k$  orthonormal. The norm of  $\mathfrak{n}$  will be the one induced by this product.

**Lemma 4.3.** *Let  $X \in \text{DC}_\tau(\mathfrak{n})$ . Let  $\mathcal{O}$  be a coadjoint orbit of maximal rank weakly integral (with respect to  $\Gamma$ ) and let  $\pi \in \Pi_{\mathcal{O}}$ . There exists a constant*

$C_\Gamma > 0$  such that for all  $f \in W^{n-1+\tau}(H_\pi)$  we have:

$$(60) \quad \delta_{\mathcal{O}}(X)^{-1} \|f\| \leq C_\Gamma \|f\|_{n-1+\tau}.$$

*Proof.* Let  $\lambda \in \mathcal{O}$ . Since  $\mathcal{O}$  is weakly integral (with respect to  $\Gamma$ ) and has maximal rank, we claim that there exist  $Y \in \mathcal{E}^{k-1}$  such that

$$(61) \quad M_Y := (B_\lambda(E_1^1, Y), \dots, B_\lambda(E_n^1, Y)) \in \mathbb{Z}^n \setminus \{0\}.$$

In fact, if  $Y \in \mathcal{E}^{k-1}$  then  $[E_j^1, Y] \in \mathfrak{n}_k$ , since  $\mathcal{E}^{k-1} \subset \mathfrak{n}_{k-1}$ . By the Baker-Campbell-Hausdorff formula, we have  $(\exp E_j^1, \exp Y) = \exp[E_j^1, Y]$ . It follows that  $[E_j^1, Y] \in \log Z(\Gamma)$  and, since  $\lambda$  is weakly integral, we have  $B_\lambda(E_j^1, Y) = \lambda([E_j^1, Y]) \in \mathbb{Z}$  for all  $j \in \{1, \dots, n\}$ . If  $B_\lambda(E_j^1, Y) = 0$ , for all  $Y \in \mathcal{E}^{k-1}$  and all  $j \in \{1, \dots, n\}$ , then  $\mathfrak{n}_{k-1}^\perp(\mathcal{O}) = \mathfrak{n}$  (see Lemma 2.2), contradicting the hypothesis that  $\lambda$  has maximal rank. The claim is proved.

Since  $Y \in \mathcal{E}^{k-1}$ , by definition we have  $\|Y\| = 1$ . Hence

$$(62) \quad \delta_{\mathcal{O}}(X) \geq |B_\lambda(X, Y)| \geq |\langle M_Y, \Omega_X \rangle| \geq \frac{K}{|M_Y|^{n-1+\tau}}.$$

Let  $\pi \in \Pi_{\mathcal{O}}$  be an irreducible representation of the Lie algebra  $\mathfrak{n}$  on the Hilbert space  $H := H_\pi$ . Let  $P = -(2\pi)^{-2} \sum_{i=1}^n [E_i, Y]^2 \in \mathfrak{U}(\mathfrak{n})$ . Since  $[E_i, Y] \in Z(\mathfrak{n})$ , for all  $i \in \{1, \dots, n\}$ , the operator  $\pi_*(P) = |M_Y|^2 \text{Id}_H$ . Hence for any  $f \in H$  we have

$$(63) \quad \delta_{\mathcal{O}}(X)^{-1} \|f\| \leq K^{-1} \| |M_Y|^{n-1+\tau} f \| = K^{-1} \| \pi(P)^{\frac{n-1+\tau}{2}} f \|.$$

Since  $P$  has order 2 as an element of the enveloping algebra  $\mathcal{U}(\mathfrak{n})$  (in fact  $P \leq (2\pi)^{-2} \Delta$ ), if  $f \in W^{n-1+\tau}(H)$ , the inequality  $\| \pi_*(P)^{\frac{n-1+\tau}{2}} f \| \leq \|f\|_{n-1+\tau}$  holds.  $\square$

**4.3. Uniform estimates.** We prove that, for any Diophantine element, appropriate Sobolev norms of the Green operators for the cohomological equation, which have been constructed in each irreducible unitary representation, are *bounded uniformly* over all irreducible unitary representations. This step is crucial in order to solve the cohomological equation on a nil-manifold  $\mathcal{N} = \Gamma \backslash N$  by "gluing" together the solutions constructed in every irreducible sub-representation of the representation of  $N$  on  $L^2(\mathcal{N}, \mu)$ .

**Lemma 4.4.** *Let  $\mathcal{O}$  be any coadjoint orbit and let  $\pi \in \Pi_{\mathcal{O}}$  be a unitary representation of  $N$ . For all  $f \in W^1(H_\pi)$ , the following inequality holds:*

$$(64) \quad w_Z(\mathcal{O}) \|f\| \leq \frac{1}{2\pi} \|f\|_1.$$

*Proof.* By the definition (36) of the non-negative number  $w_Z(\mathcal{O})$ , for each coadjoint orbit  $\mathcal{O}$  of  $N$ , there exists an element  $Z \in Z(\mathfrak{n})$  such that  $\|Z\| = 1$ , with respect to the fixed euclidean norm on  $\mathfrak{n}$ , and  $|\lambda(Z)| = w_Z(\mathcal{O})$ , for

any  $\lambda \in \mathcal{O}$ . In addition, since  $Z \in Z(\mathfrak{n})$ , for any  $\pi \in \Pi_{\mathcal{O}}$  we have that  $\pi(Z) = 2\pi\iota\lambda(Z)$ . It follows that, for any  $f \in W^1(H_{\pi})$ ,

$$(65) \quad w_Z(\mathcal{O})\|f\| = \frac{1}{2\pi} \|\pi_*(Z)f\| \leq \frac{1}{2\pi} \|f\|_1,$$

as claimed.  $\square$

**Corollary 4.5.** *Let  $X \in DC_{\tau}(\mathfrak{n})$ . Let  $\mathcal{O}$  be a non-trivial coadjoint orbit, integral with respect to a lattice  $\Gamma$ , and let  $\pi \in \Pi_{\mathcal{O}}$  an irreducible representation of the Lie algebra  $\mathfrak{n}$  on a Hilbert space  $H_{\pi}$ .*

- (1) *If  $\alpha > n + \tau - 1/2$  and  $\beta < -1/2$ , there exists  $C'_{\Gamma} := C'_{\Gamma}(X, \alpha) > 0$  such that, for all  $f \in W^{\alpha}(H_{\pi})$ ,*

$$\|G_X f\|_{\beta} \leq C'_{\Gamma} \|f\|_{\alpha};$$

- (2) *if  $f \in W^{\alpha}(H_{\pi})$ ,  $\alpha > n + \tau$ , and  $D(f) = 0$ , for all  $D \in \mathcal{I}_X^{\alpha}(H_{\pi})$ , then  $G_X f \in W^{\beta}(H_{\pi})$ , for all  $\beta < [\alpha - (n + \tau)][(n + \tau)k + 1]^{-1}$  and there exists  $C''_{\Gamma} := C''_{\Gamma}(X, \alpha, \beta) > 0$  such that*

$$\|G_X f\|_{\beta} \leq C''_{\Gamma} \|f\|_{\alpha}.$$

*Proof.* The proof is by induction on the degree of nilpotency  $k \in \mathbb{N} \setminus \{0\}$  of the nilpotent group  $N$ . If  $k = 1$  the group  $N$  is abelian, all irreducible unitary representations are one-dimensional and (weakly) integral. The Diophantine condition immediately implies that the statement holds in this case with the exception of the trivial representation. Let us assume that the statement holds for all lattices in any  $(k - 1)$ -step nilpotent group and let  $N$  be a  $k$ -step nilpotent group. If the coadjoint orbit  $\mathcal{O}$  is of maximal rank and integral with respect to a lattice  $\Gamma \subset N$ , then the statement follows from Theorem 3.6, Lemma 4.3 and Lemma 4.4.

If  $\mathcal{O}$  is integral (with respect to  $\Gamma$ ), but it is not of maximal rank, then for any  $\lambda \in \mathcal{O}$ , by Lemma 2.3 the restriction  $\lambda|_{\mathfrak{n}_k}$  is identically zero and any irreducible representation  $\pi \in \Pi_{\mathcal{O}}$  factors through an irreducible representation  $\pi'$  of the  $(k - 1)$ -step nilpotent group  $N' := N / \exp \mathfrak{n}_k$ , induced by the linear functional  $\lambda' \in (\mathfrak{n}/\mathfrak{n}_k)^*$ . Since  $\lambda \in \mathfrak{n}^*$  is integral (with respect to  $\Gamma$ ), there exists a polarizing subalgebra  $\mathfrak{m}$  such that  $(\exp(2\pi\iota\lambda), \exp \mathfrak{m})$  is a maximal integral character for  $\Gamma$ . Since  $\lambda|_{\mathfrak{n}_k} \equiv 0$ , the subalgebra  $\mathfrak{m}' := \mathfrak{m}/\mathfrak{n}_k$  is polarizing for  $\lambda'$  and  $(\exp(2\pi\iota\lambda'), \exp \mathfrak{m}')$  is a maximal integral character for the lattice  $\Gamma' := \Gamma/\Gamma \cap \exp \mathfrak{n}_k$ . By the induction hypothesis, the statement holds also in this case.  $\square$

**4.4. Proof of Theorems 1.2 and 1.3.** Let  $L^2(\mathcal{N}, \mu) = \oplus_{i \in \mathbb{N}} H_i$  be the decomposition of the space of square integrable functions on  $\mathcal{N}$  into irreducible components under the right action of  $N$ . Such a decomposition

induces a decomposition of the subspace of smooth vectors,

$$C^\infty(\mathcal{N}) = \bigoplus_{i \in \mathbb{N}} C^\infty(H_i),$$

and of the Sobolev spaces: for each  $\alpha \in \mathbb{R}$ , there is an *orthogonal* splitting

$$(66) \quad W^\alpha(\mathcal{N}) = \bigoplus_{i \in \mathbb{N}} W^\alpha(H_i).$$

Consequently, for any  $X \in \mathfrak{n}$ , the spaces  $\mathcal{I}_X(\mathcal{N}) \subset \mathcal{D}'(\mathcal{N})$  of all  $X$ -invariant distributions and the subspace  $\mathcal{I}_X^\alpha(\mathcal{N}) := \mathcal{I}_X(\mathcal{N}) \cap W^{-\alpha}(\mathcal{N})$  split as follows:

$$(67) \quad \begin{aligned} \mathcal{I}_X(\mathcal{N}) &= \bigoplus_{i \in \mathbb{N}} \mathcal{I}_X(H_i); \\ \mathcal{I}_X^\alpha(\mathcal{N}) &= \bigoplus_{i \in \mathbb{N}} \mathcal{I}_X^\alpha(H_i). \end{aligned}$$

Let  $X$  be irrational. By Lemma 3.5 and by the decompositions (67), the space  $\mathcal{I}_X(\mathcal{N})$  admits a countable basis  $\mathcal{B}_X(\mathcal{N}) \subset \mathcal{I}_X^\alpha$ , for any  $\alpha > 1/2$ . This proves Theorem 1.2

Let  $X \in \text{DC}_\tau(\mathfrak{n})$ . Then the Green operator for the cohomological equation on  $\mathcal{N}$  can be constructed as follows. Let  $\alpha > n + \tau - 1/2$  and  $\beta < -1/2$ . For any  $i \in \mathbb{N}$ , let  $G_X^{(i)}$  be the Green operator for the cohomological equation for  $X$  in the irreducible representation  $H_i$ . By Corollary 4.5, the operators  $G_X^{(i)} : W^\alpha(H_i) \rightarrow W^\beta(H_i)$  are bounded and have uniformly bounded norms, in the sense that there exists  $C'_\Gamma > 0$  such that  $\|G_X^{(i)}\|_{\alpha, \beta} \leq C'_\Gamma$ .

For each  $i \in \mathbb{N}$  and  $\alpha \in \mathbb{R}$ , let  $p_\alpha^{(i)} : W^\alpha(\mathcal{N}) \rightarrow W^\alpha(H_i)$  be the orthogonal projection and  $j_\alpha^{(i)} : W^\alpha(H_i) \rightarrow W^\alpha(\mathcal{N})$  the embedding, determined by the orthogonal splitting (66). The operator  $G_X : W^\alpha(\mathcal{N}) \rightarrow W^\beta(\mathcal{N})$  defined as

$$G_X := \bigoplus_{i \in \mathbb{N}} j_\beta^{(i)} \circ G_X^{(i)} \circ p_\alpha^{(i)}$$

is a well-defined, bounded Green operator for the cohomological equation for the Diophantine vector field  $X \in \text{DC}_\tau(\mathfrak{n})$  on  $\mathcal{N}$ . In fact, by the orthogonality of the splitting (66),

$$\|G_X\|_{\alpha, \beta}^2 \leq \sum_{i \in \mathbb{N}} \|j_\beta^{(i)} \circ G_X^{(i)} \circ p_\alpha^{(i)}\|_{\alpha, \beta}^2 \leq \max_{i \in \mathbb{N}} \|G_X^{(i)}\|_{\alpha, \beta}^2 \leq (C'_\Gamma)^2.$$

Finally, if  $\alpha > n + \tau$  and  $f \in W^\alpha(\mathcal{N})$  belongs to the kernel of all  $X$ -invariant distributions  $D \in \mathcal{I}_X^\alpha(\mathcal{N})$ , then  $u := G_X(f) \in W^\beta(\mathcal{N})$  for any

$\beta < [\alpha - (n + \tau)][(n + \tau)k + 1]^{-1}$ . By construction,

$$u = \sum_{i \in \mathbb{N}} j_{\beta}^{(i)} \circ G_X^{(i)}(p_{\alpha}^{(i)}(f)) .$$

By the splitting (39), it follows that  $D_i(p_{\alpha}^{(i)}(f)) = 0$ , for all  $D_i \in \mathcal{I}_X^{\alpha}(H_i)$  and for each  $i \in \mathbb{N}$ , since  $D(f) = 0$  for all  $D \in \mathcal{I}_X^{\alpha}(\mathcal{N})$ . Hence, by Corollary 4.5, there exists a constant  $C_{\Gamma}'' > 0$  such that, for all  $i \in \mathbb{N}$ ,

$$\|G_X^{(i)}(p_{\alpha}^{(i)}(f))\|_{\beta} \leq C_{\Gamma}'' \|p_{\alpha}^{(i)}(f)\|_{\alpha} .$$

Again by the orthogonality of the splitting (66), we have

$$\|u\|_{\beta}^2 = \sum_{i \in \mathbb{N}} \|G_X^{(i)}(p_{\alpha}^{(i)}(f))\|_{\beta}^2 \leq (C_{\Gamma}'')^2 \sum_{i \in \mathbb{N}} \|p_{\alpha}^{(i)}(f)\|_{\alpha}^2 \leq (C_{\Gamma}'')^2 \|f\|_{\alpha}^2 .$$

The proof Theorem 1.3 is concluded.

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MATHÉMATIQUES, UNIVERSITÉ DE LILLE 1 (USTL), F59655 VILLENEUVE D’ASQ  
CEDEX, FRANCE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ON M5S  
2E4 CANADA

*E-mail address:* livio.flaminio@math.univ-lille1.fr

*E-mail address:* forni@math.toronto.edu